CHAOS AND THE MAP $x \mapsto \omega(x, f)$

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Abstract. Let $\mathcal{K}(2^{\mathbb{N}})$ be the class of compact subsets of the Cantor space $2^{\mathbb{N}}$, furnished with the Hausdorff metric. Let $f \in C(2^{\mathbb{N}})$. We study the map $\omega_f : 2^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ defined as $\omega_f(x) = \omega(x, f)$, the ω-limit set of x under f. Unlike the case of n-dimenisional manifolds ($n \geq 1$), we show that ω_f is continuous for the generic self-map f of the Cantor space, even though the set of functions for which ω_f is everywhere discontinuous on a subsystem is dense in $C(2^{\mathbb{N}})$. The relationships between the continuity of ω_f and three forms of chaos are investigated.

1. Introduction

One finds in the literature a variety of definitions of chaos for continuous self-maps of a compact space. Common to all, however, is the idea that points arbitrarily close together can have orbits or ω-limit sets that spread out or are far apart. For example, the notions of topological entropy and Li-Yorke chaos make explicit use of the separation of trajectories, while Devaney chaos incorporates sensitivity as well as the requirement that nearby points must generate distinct ω-limit sets. These ideas are developed at some length in [14], [21] and [7].

A notion of chaos developed by Bruckner and Ceder for continuous self-maps of the interval focuses instead on the behaviour of the map $\omega_f : I \to \mathcal{K}(I)$ that takes a point x in I to its ω-limit set $\omega(x, f)$ [9]. There, they find that the Baire class of ω_f determines quite a lot of the possible dynamical behaviour of the function f. In particular, ω_f is rarely continuous, ω_f is always in the second Baire class, and ω_f in the first Baire class determines a notion of non-chaos strictly intermediate to positive topological entropy and Li-Yorke chaos. That is,

$$h(f) > 0 \Rightarrow \omega_f \textit{ is not of Baire class 1} \Rightarrow f \textit{ is Li-Yorke chaotic},$$

but none of the reverse implications is true. Bruckner and Ceder also characterize the Baire class of the map ω_f in terms of the structure of the ω-limit sets generated by f.

In [25], Yano studies the generic properties of continuous self-maps of manifolds with dimension at least 1. As Yano shows that the typical map f in $C(M)$ possesses a horseshoe like structure K, we can conclude that positive topological entropy, Devaney chaos on the subsystem K, and Li-Yorke chaos are all present on a dense and open set of functions f in $C(M)$. Conversely, when one considers the Cantor space $2^{\mathbb{N}}$, [13] D'Aniello and Darji show that the typical element f in $C(2^{\mathbb{N}})$ has

Date: May 29, 2012.

1991 *Mathematics Subject Classification.* Primary: 37B99, 54H20; Secondary: 37B10.

Key words and phrases. Cantor space; continuous self-map; ω-limit set; chaos.

This research has been partially supported by “Gruppo Nazionale per l'Analisi Matematica, la Probabilità e le loro Applicazioni dell'Istituto Nazionale di Alta Matematica F.Severi”.

zero topological entropy and is Devaney chaotic on no subsystem K contained in $2^{\mathbb{N}}$. Most recently, [3] Bernardes and Darji establish that the typical f in $C(2^{\mathbb{N}})$ has no Li-Yorke pair.

Our work can be viewed as a continuation of that of [9], [25], [13] and [3].

After establishing definitions, notation and other background material in section 2, we develop our main results in section 3. In particular, we show that for a dense set of functions f in $C(2^{\mathbb{N}})$, the map ω_f is everywhere discontinuous on a subsystem $K \subseteq 2^{\mathbb{N}}$. Nevertheless, for a typical function f in $C(2^{\mathbb{N}})$ the map $\omega_f : 2^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ is continuous. We also give a direct proof showing that the typical f in $C(2^{\mathbb{N}})$ has no Li-Yorke pair, providing the results of [13] and of [3] mentioned above as corollaries.

In section 4 we study the behaviour of the map $\omega_f : X \to \mathcal{K}(X)$ when the topological system (X, f) is Devaney chaotic. There, we find that there are only two possibilities: either ω_f is everywhere discontinuous, or ω_f is continuous precisely on the set $\{x \in X : \omega(x, f) = X\}$.

The principal result of section 5 shows that, for any subsystem (K, σ) of the shift map σ on the sequence space Σ_n formed by n symbols, the map $\omega_\sigma : K \to \mathcal{K}(\Sigma_n)$ is everywhere discontinuous. This result is necessary in section 6, where various examples are presented to illustrate the relationship between the behaviour of ω_f, and the chaotic nature of $f : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$.

2. Definitions and background material

Definition 2.1. *A topological dynamical system (X, f) is a compact metric space X and a continuous self-map f of X.*

We denote the space $C(X, X)$ comprised of all continuous maps from X into X, simply by $C(X)$. We endow this set with the sup norm and hence it forms a complete and separable metric space. We also consider $\mathcal{K}(X)$, the space of all non-empty closed subsets of X, endowed with the Hausdorff metric $\mathcal{H}$. This space $(\mathcal{K}(X), \mathcal{H})$ is compact.

Let Y be a complete metric space. We say that *a generic (or a typical) element of Y has property P* if the set of elements of Y which have property P contains a dense G_δ subset of Y. Equivalently, the set of those elements of Y which do not satisfy property P is a *meager subset of Y*, i.e., it is the union of countably many nowhere dense sets [23].

A topological space is a *Cantor space* if it is homeomorphic to the middle $\frac{1}{3}$-Cantor subset of the unit interval. Every non-empty, compact, perfect, zero-dimensional metric space is a Cantor space. Our model of the Cantor space, in this paper, is ${\Sigma_2}^{\mathbb{N}} = \{0,1\}^{\mathbb{N}} = 2^{\mathbb{N}}$, the set of all infinite sequences of 0's and 1's. We use $(\alpha_1, \alpha_2, \dots)$ to denote an element of $2^{\mathbb{N}}$. We denote by σ the *shift map*, that is $\sigma : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ is defined by

$$\sigma(\alpha_1, \alpha_2, \dots) = (\alpha_2, \alpha_3, \dots).$$

We use $\alpha = (\alpha(1), \alpha(2), \dots)$ to denote an element of $(2^{\mathbb{N}})^{\mathbb{N}}$.

Positive topological entropy

The definition we use is due to Bowen [10] and Dinaberg [15].

Let (X, d) be a compact metric space and $f \in C(X)$. For each $n \in \mathbb{N}$, define on X the metric d_n as

$$d_n(x, y) = \max\{d(f^i(x), f^i(y)) : 0 \leq i < n\}.$$

A subset E of X is said to be (n, ϵ, f)*-separated* if each pair of distinct points of E is at least ϵ apart in the metric d_n. Denote by $N(n, \epsilon, f)$ the maximum cardinality of an (n, ϵ, f)-separated set. The *topological entropy of the map* f is defined by

$$ent(f) = \lim_{\epsilon \to 0} \left(\limsup_{n \to \infty} \frac{1}{n} \log N(n, \epsilon, f) \right).$$

Among the many important properties of the topological entropy, there is the fact that topological entropy is an invariant of topological conjugacy. The shift map has entropy equal to $\ln 2$. For more information on topological entropy see [11] .

Devaney chaos

Let (X, d) be a compact metric space without isolated points and $f \in C(X)$. A point $x \in X$ is *a transitive point of* f if its trajectory $\gamma(x, f) = \{x, f(x), f^2(x), \dots\}$ is dense in X. The function f is said to be *transitive* if it has a transitive point. We say that the system (X, f) is *Devaney chaotic* if the following conditions hold:

(1) (X, f) is transitive;
(2) the set of periodic points of f is dense in X;
(3) f has a sensitive dependence on initial conditions, i.e., there is $s > 0$ such that for all $x \in X$ and $\epsilon > 0$, there exist $y \in X$ within ϵ of x and $n \in \mathbb{N}$ such that $d(f^n(x), f^n(y))$, the distance between $f^n(x)$ and $f^n(y)$, is greater than s.

It is a well-known result that conditions (1) and (2) imply (3). For example, see [2], or [16], or [1]. We will say that (X, f) *is Devaney chaotic on a subsystem* if there is a perfect subset K of X such that $f(K) = K$ and $f|_K$, the restriction of f to K, is Devaney chaotic. The shift map is Devaney chaotic. For example, see [13]. For more information on transitivity and on Devaney chaos we refer the reader to [26] and [14].

Li-Yorke chaos

A function f is *chaotic* (in the sense of Li and Yorke) if there exists an uncountable set S such that

$$\limsup_{n \to \infty} |f^n(x) - f^n(y)| > 0$$

and

$$\liminf_{n \to \infty} |f^n(x) - f^n(y)| = 0$$

for each distinct x and y in S [21].

Devaney chaos implies Li-Yorke chaos ([18]; [20]; [22]) and positive topological entropy implies Li-Yorke chaos ([5]). If X is the unit interval we have that positive topological entropy is equivalent to Devaney chaos on a subsystem. In the general setting of compact metric spaces there are no implications between positive topological entropy and Devaney chaos on a subsystem ([24], [19], [6]). See [4] for a good discussion of the relationships that exist between various notions of chaos defined on compact spaces.

Bruckner-Ceder chaos on the unit interval

Let $f \in C(I)$, and consider $\{\omega(x, f) : x \in I\}$, the family of ω-limit sets of f, endowed with the Hausdorff metric. This spaces is compact [8]. Bruckner and Ceder in [9] ask questions related to the continuity of the map $\omega_f : x \mapsto \omega(x, f)$. They find that ω_f is rarely continuous. It is always in (at most) the second Baire class. In particular they prove that

$$h(f) > 0 \Rightarrow \omega_f \textit{ is not of Baire class 1} \Rightarrow f \textit{ is Li-Yorke chaotic},$$

with none of these implications being reversable.

Theorem 2.2. *Let M be a compact n-dimensional manifold with $n \geq 1$. The set*

$$\mathcal{BCC} = \{f \in C(M) : \omega_f \textit{ is everywhere discontinuous on a subsystem}\}$$

is dense and open.

The proof of Theorem 2.2 rests on the existence of an horseshoe for a dense and open set of functions f in $C(M)$. In particular, there exists a subsystem K topologically conjugate to the shift map on a finite number of symbols. Thus, $h(f_{|K}) > 0$, $f_{|K}$ is Devaney chaotic and $\omega_f : K \to \mathcal{K}(K)$ is everywhere discontinuous [25].

Theorem 2.3. [13] *The generic $f \in C(2^{\mathbb{N}})$ has zero topological entropy and it is Devaney chaotic on no subsystem.*

3. Main results

Our primary interest is the behaviour of the map taking x to its ω-limit set $\omega(x, f)$. We begin by showing that $\omega_f : X \to \mathcal{K}(X)$ always has a certain degree of regularity, as reflected by the Borel class of ω_f. Theorem 3.1 is the obvious analogue of Theorem 2.4 in [9], where the authors restrict their attention to $X = [0, 1]$.

Theorem 3.1. *Let X be a compact metric space and let $f \in C(X)$. Then ω_f is of Borel class 2.*

Proof. Let K be a compact set and $\{a_i : i = 1, 2, \dots\}$ be a countable dense subset of K. Let $\epsilon > 0$. It will suffice to show that $\{x : \mathcal{H}(\omega(x, f), K) < \epsilon\}$ is a $G_{\delta\sigma}$ set. For any C let $S_\epsilon(C) = \{y : d(z, y) < \epsilon \textit{ for some } z \in C\}$. Put $A = \{x : \omega(x, f) \subseteq S_\epsilon(K)\}$ and $B = \{x : K \subseteq S_\epsilon(\omega(x, f))\}$. By definition $\mathcal{H}(\omega(x, f), K) < \epsilon$ if and only if $x \in A \cap B$. It is easily verified that

$$A = \cup_{n=1}^{\infty} \cup_{m=1}^{\infty} \cap_{k=m}^{\infty} \{x : d(f^k(x), K) \leq \frac{n\epsilon}{n+1}\}$$

and

$$B = \cup_{n=1}^{\infty} \cap_{j=1}^{\infty} \cap_{m=1}^{\infty} \cup_{k=m}^{\infty} \{x : d(f^k(x), a_j) < \frac{n\epsilon}{n+1}\}.$$

Clearly, A is an F_σ and B is a $G_{\delta\sigma}$ set so that $A \cap B$ is a $G_{\delta\sigma}$ set, completing the proof.

$\square$

Theorem 3.2. *The set*

$$\mathcal{BCC} = \{f \in C(2^{\mathbb{N}}) : \omega_f \textit{ is everywhere discontinuous on a subsystem}\}$$

is dense in $C(2^{\mathbb{N}})$.

Proof. Let $g \in C(2^{\mathbb{N}})$ and $\epsilon > 0$. We will show that $B_\epsilon(g) \cap \mathcal{BCC}$ is non-empty.
Let $f : \left(2^{\mathbb{N}}\right)^{\mathbb{N}} \to \left(2^{\mathbb{N}}\right)^{\mathbb{N}}$ be defined as

$$f(\alpha(1), \alpha(2), \dots) = (\sigma(\alpha(1)), \sigma(\alpha(2)), \dots),$$

and $F : \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}} \to \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ defined by $F(i, \alpha) = ((i+1) \bmod n, f(\alpha))$. As proved in [13], F has infinite topological entropy and is Devaney chaotic.
By Lemma 3.3 in [12] we may choose $g_1 \in B_{\frac{\epsilon}{3}}(g)$ such that the orbit of some point of $2^{\mathbb{N}}$ under g_1 is finite. Applying Lemma 3.4 of [12] to this g_1, we may choose $g_2 \in B_{\frac{\epsilon}{3}}(g_1)$ and pairwise disjoint non-empty clopen subsets $V_0, \dots, V_{n-1}$ of $2^{\mathbb{N}}$ such that for all $0 \le i \le n-1$, we have that

(1) $g_2(V_i) \subseteq V_{(i+1) \bmod n}$
(2) $diam(V_i) < \frac{\epsilon}{3}$.

As in [13] we define g_3, a modification of g_2 on $\cup_{i=0}^{n-1} V_i$, by setting

$$g_3(x) = \begin{cases} g_2(x) & \text{if } x \in 2^{\mathbb{N}} \setminus \cup_{i=0}^{n-1} V_i \\ h^{-1} \circ F \circ h(x) & \text{if } x \in \cup_{i=0}^{n-1} V_i. \end{cases}$$

Then, $g_3(V_i) \subseteq V_{(i+1) \bmod n}$ and hence $d(g_3, g_2) < \frac{\epsilon}{3}$. This, in turn, implies that $d(g_3, g) < \epsilon$. Furthermore, the restriction of g_3 to the union of $\cup_{i=0}^{n-1} V_i$ is topologically conjugate to F.
We now show that $\omega_F : \{0, 1, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}} \to \mathcal{K}(\{0, 1, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}})$ is discontinuous everywhere.
Let $X = \{0, 1, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$. It suffices to show that the sets $\{z \in X : \omega_F(z) = X\}$ and $\{z \in X : \omega_F(z) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}}\}$, where by $\{\overline{0}\}$ we denote the infinite sequence with all the entries equal to 0, are both dense in X. As F is transitive, the set $\{z \in X : \omega_F(z) = X\}$ is, in fact, dense in X. Now, let $x \in X$, with $\epsilon > 0$. It suffices to find $\tilde{x}$ in X such that $d(x, \tilde{x}) < \epsilon$ and $\omega_F(\tilde{x}) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}}$ in order to show that $\{z \in X : \omega_F(z) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}}\}$ is dense in X. Set $x = \{i\} \times (\alpha(1), \alpha(2), \alpha(3), \dots)$. Following [13], there exists a subset $U \subseteq B_\epsilon(x)$ of the form $\{i\} \times ([\alpha(1)] \times [\alpha(2)] \times \dots \times [\alpha(k)] \times 2^{\mathbb{N}} \times 2^{\mathbb{N}} \times \dots)$, where $[\alpha(1)]$, $\dots$, $[\alpha(k)]$ are sufficiently long initial segments of $\alpha(1), \dots, \alpha(k)$, respectively. Now, set $\tilde{x} = \{i\} \times ([\alpha(1)]\overline{0} \times [\alpha(2)]\overline{0} \times \dots \times [\alpha(k)]\overline{0} \times \overline{0} \times \overline{0} \times \overline{0} \times \dots)$ so that $\tilde{x} \in U \subseteq B_\epsilon(x)$. Since $\alpha(j)$, for $1 \le j \le k$, terminates in a tail of zeroes, we see that $\omega_F(\tilde{x}) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}}$. In order to complete the proof of our theorem, it suffices to show that the map g_3 has the property that $\omega_{g_3} : \cup_{i=0}^{n-1} V_i \to \mathcal{K}(\cup_{i=0}^{n-1} V_i)$ is everywhere discontinuous. Recall that $g_3 : \cup_{i=0}^{n-1} V_i \to \cup_{i=0}^{n-1} V_i$ is topologically conjugate to the map $F : X \to X$, so that $g_3(x) = (h^{-1} \circ F \circ h)(x)$, and the following diagram commutes

$$\begin{array}{ccc} \cup_{i=0}^{n-1} V_i & \xrightarrow{g_3} & \cup_{i=0}^{n-1} V_i \\ \downarrow h & & \downarrow h^{-1} \\ X & \xrightarrow{F} & X \end{array}$$

Since $A = \{z \in X : \omega_F(z) = X\}$ is dense in X, and $h^{-1} : X \to \cup_{i=0}^{n-1} V_i$ is a homeomorphism, it folllows that $h^{-1}(A) = \{z \in \cup_{i=0}^{n-1} V_i : \omega_{g_3}(z) = \cup_{i=0}^{n-1} V_i\}$ is dense in $\cup_{i=0}^{n-1} V_i$. Similarly, since $B = \{z \in X : \omega_F(z) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}}\}$ is dense in X, it follows that $h^{-1}(B) = \{z \in \cup_{i=0}^{n-1} V_i : \omega_{g_3}(z) = h^{-1}(\{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}})$ is also dense in $\cup_{i=0}^{n-1} V_i$. □

Theorem 3.3. *The set $\mathcal{C} = \{f \in C(2^{\mathbb{N}}) : \omega_f : 2^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ is continuous$\}$ is a dense set of type G_δ in $C(2^{\mathbb{N}})$.*

Proof. Since the set of points of continuity for any map between two metric spaces is of type G_δ, it suffices to show that $\mathcal{C}$ is dense in $\mathcal{C}(2^{\mathbb{N}})$. Let $f \in \mathcal{C}(2^{\mathbb{N}})$, with $\epsilon > 0$. We find g in $C(2^{\mathbb{N}})$ so that $g \in B_\epsilon(f)$ and with the property that for any $x \in 2^{\mathbb{N}}$ there exists an open set U containing x such that $\omega_{g_{|U}}$ is a constant.

Since f is continuous on $2^{\mathbb{N}}$ compact, f is uniformly continuous there. Take $0 < \eta < \epsilon$ so that $|f(x) - f(y)| < \epsilon$ whenever x and y are in $2^{\mathbb{N}}$, and $|x - y| < \eta$. Now, let $U_1, \dots, U_m$ be pairwise disjoint clopen subsets of $2^{\mathbb{N}}$ so that $\cup_{i=1}^m U_i = 2^{\mathbb{N}}$, and $diam(U_i) < \eta$ for all i. For each i, fix $x_i \in U_i$. We define $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ so that $g_{|U_i} = f(x_i) = y_i$.

Since $|f(x) - f(y)| < \epsilon$ whenever x and y are contained in U_i, it follows that $g \in B_\epsilon(f)$. If $x \in 2^{\mathbb{N}}$, then there exists $i \in \{1, 2, \dots, m\}$ so that $x \in U_i$. The set U_i is open, and since $g(U_i) = y_i$ is a constant, it follows that $\omega(z, g) = \omega(y_i, g)$ for all $z \in U_i$.

□

Theorem 3.4 reproduces a result found in [3]. We include it since our proof is direct and involves only minor modification of that for Theorem 3.3.

Theorem 3.4. *The set $\mathcal{NLY} = \{f \in C(2^{\mathbb{N}}) : f$ has no Li-Yorke pair$\}$ is dense and of type G_δ.*

Proof. Let G_m be the collection of functions f in $C(2^{\mathbb{N}})$ such that for any pair of points $\{x, y\}$ in $2^{\mathbb{N}}$, if

$$\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0,$$

then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| < \frac{1}{m}.$$

It suffices to show that G_m contains a dense and open subset of $C(2^{\mathbb{N}})$. In particular, if $f \in \cap_{m=1}^\infty G_m$, and $\{x, y\} \subset 2^{\mathbb{N}}$ so that $\liminf_{n\to\infty} |f^n(x) - f^n(y)| = 0$, then

$$\limsup_{n\to\infty} |f^n(x) - f^n(y)| < \frac{1}{m},$$

for all m in $\mathbb{N}$, and

$$\lim_{n\to\infty} |f^n(x) - f^n(y)| = 0.$$

Thus, if $f \in \cap_{m=1}^\infty G_m$, then f has no Li-Yorke pair.

We first show that G_m is dense. Let $f \in C(2^{\mathbb{N}})$ with $\epsilon > 0$ and $m \in \mathbb{N}$. Since f is

continuous on $2^{\mathbb{N}}$ compact, there exists $0 < \eta < \min\{\epsilon, \frac{1}{m}\}$ so that $|f(x) - f(y)| < \min\{\epsilon, \frac{1}{m}\}$ whenever $|x - y| < \eta$.
Let $U_1, U_2, \ldots, U_k$ be pairwise disjoint clopen subsets of $2^{\mathbb{N}}$ so that $\cup_{i=1}^{k} U_i = 2^{\mathbb{N}}$ and $diam(U_i) < \eta$ for all i. For each i, fix $x_i \in U_i$. We define $g : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ so that $g_{|U_i} = f(x_i) = y_i$. Since $|f(x) - f(y)| < \epsilon$ whenever x and y are contained in U_i, it follows that $g \in B_\epsilon(f)$. Now, suppose that $\{x, y\} \subset 2^{\mathbb{N}}$ and $\liminf_{n\to\infty} |g^n(x) - g^n(y)| = 0$. There exists $j \in \mathbb{N}$ so that $g^j(x)$, $g^j(y)$ are both contained in the same element of the partition $\{U_1, \ldots, U_k\}$, say $\{g^j(x), g^j(y)\} \subseteq U_{k(j)}$. It follows, then, that $g(g^j(x)) = g(g^j(y)) = y_{k(j)}$, and $\lim_{n\to\infty} |g^n(x) - g^n(y)| = 0$. We conclude that G_m is dense.
We now show that there exists some $\gamma > 0$ so that $B_\gamma(g) \subseteq G_m$. Consider the set $\{y_1, y_2, \ldots, y_k\}$. Each y_j is contained in some one element of the partition $\{U_1, U_2, \ldots, U_k\}$ of $2^{\mathbb{N}}$. Say $y_j \in U_{k(j)}$. Since each U_i is open, for each y_j there exists $\gamma_j > 0$ so that $B_{\gamma_j}(y) \subseteq U_{k(j)}$. Let $\gamma = \min\{\gamma_1, \gamma_2, \ldots, \gamma_k\}$. We show that $B_\gamma(g) \subseteq G_m$. Let $h \in B_\gamma(g)$, and suppose that $\liminf_{n\to\infty} |h^n(x) - h^n(y)| = 0$. As before, there exist $j \in \mathbb{N}$ and $k(j) \in \{1, \ldots, k\}$ so that $\{h^j(x), h^j(y)\} \subseteq U_{k(j)}$. Thus, for any $l > j$, it follows from our choice of γ that $\{h^l(x), h^l(y)\}$ will always be contained in some $B_\gamma(y_i) \subseteq U_{k(i)}$. Since $diam(U_i) < \frac{1}{m}$ for all i, it follows that $\limsup_{n\to\infty} |h^n(x) - h^n(y)| < \frac{1}{m}$.

□

Corollary 3.5. *The generic $f \in C(2^{\mathbb{N}})$ has zero topological entropy and it is Devaney chaotic on no subsystem.*

4. Devaney chaos and ω_f

Definition 4.1. *Let $f : X \to Y$ be a function from a topological space X into a metric space Y. The oscillation of f is defined at each $x \in X$ by*

$$O_f(x) = inf\{diam(f(U)) : U \text{ is a neighborhood of } x\}.$$

Remark 4.2. *Let $f : X \to Y$ be a function from a topological space X into a metric space Y, with $x_0 \in X$. Then, the following are equivalent:*

1. *f is continuous at x_0,*
2. *$O_f(x_0) = 0$,*
3. *for any $\epsilon > 0$ there exists $\delta > 0$ so that $O_f(B_\delta(x_0)) < \epsilon$.*

Proposition 4.3. *Let $f \in C(X)$, $\epsilon > 0$, and suppose that there exists $\delta > 0$ so that $O_{\omega_f}(B_\delta(x_0)) < \epsilon$. Then $O_{\omega_f}(\cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))) < \epsilon$.*

Proof. Let $\{y, z\} \subseteq \cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))$. Then there exist k, l in $\mathbb{N}$ so that $f^k(y) \in B_\delta(x_0)$ and $f^l(z) \in B_\delta(x_0)$. Then $\mathcal{H}(\omega(y, f), \omega(z, f)) = \mathcal{H}(\omega(f^k(y), f), \omega(f^l(z), f)) < \epsilon$. □

Let $f \in C(X)$, with $K \subseteq X$ such that $f_{|K}$ is Devaney chaotic. Define $\mathcal{S}_{f,K} = \{x \in K : \omega(x, f) = K\}$ and $\mathcal{C}_{\omega_f,K} = \{x \in K : \omega_f \textit{ is continuous at } x\}$.

Lemma 4.4. *Let $f \in C(X)$, with $K \subseteq X$ such that $f_{|K}$ is Devaney chaotic. If $\omega_f : K \to \mathcal{K}(X)$ has a point of continuity, then for any $\epsilon > 0$ there exists a dense open set $\mathcal{I}_\epsilon$ in K such that $O_{\omega_f}(\mathcal{I}_\epsilon) < \epsilon$.*

Proof. If $x \in \mathcal{S}_{f,K}$, then $\{f^n(x)\}_{n=1}^{\infty}$ is dense in K, so that for any x_0 in K and any $\delta > 0$, there exists $m \in \mathbb{N}$ so that $f^m(x) \in B_\delta(x_0)$. Let x_0 be a point of continuity of $\omega_f : K \to \mathcal{K}(X)$, with $\delta > 0$ so that $O_{\omega_f}(B_\delta(x_0)) < \epsilon$. Since $x \in \cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))$, it follows that $\mathcal{S}_{f,K} \subseteq \cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))$. Since $\mathcal{S}_{f,K}$ is dense in K, and $\cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))$ is open in K, our conclusion follows with $\mathcal{I}_\epsilon = \cup_{n=0}^{\infty} f^{-n}(B_\delta(x_0))$. □

Corollary 4.5. *Let $f \in C(X)$, with $K \subseteq X$ such that $f_{|K}$ is Devaney chaotic. If $\omega_f : K \to \mathcal{K}(X)$ has a point of continuity, then $\mathcal{S}_{f,K} \subseteq \mathcal{I}_\epsilon$, for any $\epsilon > 0$.*

Corollary 4.6. *Let $f \in C(X)$, with $K \subseteq X$ such that $f_{|K}$ is Devaney chaotic. If $\omega_f : K \to \mathcal{K}(X)$ is continuous at some point x_0 in K, then $\mathcal{C}_{\omega_f,K} = \mathcal{S}_{f,K}$.*

Proof. First, observe that $\mathcal{C}_{\omega_f,K} = \cap_{n=1}^{\infty} \mathcal{I}_{\frac{1}{n}}$. Since $\mathcal{S}_{f,K} \subseteq \mathcal{I}_{\frac{1}{n}}$ for all n in $\mathbb{N}$, it follows that $\mathcal{S}_{f,K} \subseteq \mathcal{C}_{\omega_f,K}$. If $x \notin \mathcal{S}_{f,K}$, then $\omega(x,f) \neq K$, so that $x \notin \mathcal{C}_{\omega_f,K}$. This follows from the observation that $\overline{\mathcal{S}_{f,K}} = K$. □

Corollary 4.7. *Let $f \in C(X)$ with $K \subseteq X$, such that $f_{|K}$ is Devaney chaotic. Then either*

(1) $\mathcal{C}_{\omega_f,K} = \emptyset$, *or*
(2) $\mathcal{C}_{\omega_f,K} = \{x \in K : \omega(x,f) = K\}$.

5. A result concerning shift maps

Let $\Sigma_n = \{0, 1, \dots, n-1\}^{\mathbb{N}}$, and if $x = x_0 x_1 \dots$ and $y = y_0 y_1 \dots$, set $d(x,y) = \sum_{k=0}^{\infty} \frac{|x_k - y_k|}{n^k}$.
We let

$$B_{b_0 b_1 \dots b_{m-1}} = \{x \in \Sigma_n : x_0 x_1 \dots x_{m-1} = b_0 b_1 \dots b_{m-1}\}.$$

Definition 5.1. *Call $B_{b_0 b_1 \dots b_{m-1}}$ the m-block determined by $b_0 b_1 \dots b_{m-1}$.*

Lemma 5.2. *The set $B_{b_0 b_1 \dots b_{m-1}}$ is open in Σ_n.*

Proof. Set $\epsilon = \frac{1}{n^{m-1}}$. If $d(x,y) < \frac{1}{n^{m-1}}$, then $x_i = y_i$, for $0 \le i \le m-1$. Thus

$$x \in B_{b_0 b_1 \dots b_{m-1}} \Rightarrow y \in B_{b_0 b_1 \dots b_{m-1}}.$$

Hence $B_\epsilon(x) \subseteq B_{b_0 b_1 \dots b_{m-1}}$ whenever $x \in B_{b_0 b_1 \dots b_{m-1}}$. It follows that every point in $B_{b_0 b_1 \dots b_{m-1}}$ is interior, so that $B_{b_0 b_1 \dots b_{m-1}}$ is open. □

Corollary 5.3. *The set $\cup_{k=0}^{\infty} \sigma^{-k}(B_{b_0 b_1 \dots b_{m-1}})$ is open in Σ_n.*

Proposition 5.4. *The following holds:*
$z \in \cup_{k=0}^{\infty} \sigma^{-k}(B_{b_0 b_1 \dots b_{m-1}})$ if and only if $z = z_0 z_1 z_2 \dots$ somewhere contains the block $b_0 b_1 \dots b_{m-1}$.

Proof. Recall that σ is the shift map on n symbols. If $z \in \cup_{k=0}^{\infty} \sigma^{-k}(B)$, then there exist $x \in B_{b_0 b_1 \dots b_{m-1}}$ and $k \in \mathbb{N}$ so that $\sigma^k(z) = x$. Since $x_0 x_1 \dots x_{m-1} = b_0 b_1 \dots b_{m-1}$, it follows that $z = z_0 z_1 \dots z_{k-1} b_0 b_1 \dots b_{m-1} \dots$. Suppose now that $z = z_0 \dots z_{p-1} b_0 b_1 \dots b_{m-1} \dots$. Then, $\sigma^p(z) = b_0 b_1 \dots b_{m-1}$, so that $\sigma^p(z) \in B_{b_0 b_1 \dots b_{m-1}}$, and $z \in \cup_{k=0}^{\infty} \sigma^{-k}(B_{b_0 b_1 \dots b_{m-1}})$. □

Remark 5.5. *If $K = \Sigma_n$, then $\omega_\sigma : K \to \mathcal{K}(\Sigma_n)$ is nowhere continuous.*

Lemma 5.6. *If $K \subseteq \Sigma_n$ and $\sigma_{|K}$ is Devaney chaotic, then K is nowhere dense and perfect in Σ_n.*

Proof. The set K is closed since $K = \omega(x, \sigma)$ for some $x \in K$. Moreover, K is perfect since both $\mathcal{S}_{\sigma,K}$ and $P(\sigma) \cap K$ are dense in K. If K is somewhere dense in Σ_n, then it follows that $K = \Sigma_n$. $\square$

Proposition 5.7. *The collection of all m-blocks, where m ranges through $\mathbb{N}$, is a basis for Σ_n.*

Proof. Suppose $\mathcal{I}$ is open in Σ_n, and $x \in \mathcal{I}$. Then, there exists $\epsilon > 0$ so that $B_\epsilon(x) \subseteq \mathcal{I}$. Take $k \in \mathbb{N}$ so that $\frac{1}{n^{k-1}} < \epsilon$. If $y \in B_{x_0 x_1 \ldots x_{k-1}}$, then

$$\begin{aligned} d(x,y) &= \sum_{i=0}^{\infty} \frac{|x_i - y_i|}{n^i} = \sum_{i=k}^{\infty} \frac{|x_i - y_i|}{n^i} \\ &\leq \sum_{i=k}^{\infty} \frac{n-1}{n^i} = \frac{n-1}{n^k} \sum_{i=0}^{\infty} \frac{1}{n^i} = \frac{n-1}{n^k} \frac{n}{n-1} \\ &= \frac{n}{n^k} = \frac{1}{n^{k-1}} < \epsilon. \end{aligned}$$

$\square$

Theorem 5.8. *Let $K \subseteq \Sigma_n$ such that $\sigma_{|K}$ is Devaney chaotic. Then, there exists an at most countable collection of blocks B_0, B_1, B_2, $\ldots$ so that $K = \Sigma_n \setminus \cup_{i=0}^{\infty} \cup_{k=0}^{\infty} \sigma^{-k}(B_i)$.*

Proof. This follows from Lemma 5.6 and Proposition 5.7. We need only add the observation that if $G \subseteq \Sigma_n \setminus K$, then $\sigma^{-1}(G) \subseteq \Sigma_n \setminus K$.

$\square$

Corollary 5.9. *The following holds:*
$z \in K = \Sigma_n \setminus \cup_{i=0}^{\infty} \cup_{k=0}^{\infty} \sigma^{-k}(B_i)$ if and only if $z = z_0 z_1 z_2 \ldots$ contains no B_i block, for any i.

Proof. $(\Rightarrow)$:
Suppose z contains the B_i block, for some i. Say $z = z_0 \ldots z_{p-1} b_0 \ldots b_{m-1} \ldots$, where $b_0 \ldots b_{m-1}$ is the B_i block. Then $\sigma^p(z) = b_0 \ldots b_{m-1} \ldots$ is contained in the B_i block , so that $z \in \Sigma_n \setminus K$.
$(\Leftarrow)$:
Now, suppose z contains no B_i block, for any i. In particular, if z does not contain the fixed B_i block, then $z \notin \cup_{k=0}^{\infty} \sigma^{-k}(B_i)$. Thus, $z \in \Sigma_n \setminus \cup_{k=0}^{\infty} \sigma^{-k}(B_i)$ closed for any i, so that $z \in \cap_{i=0}^{\infty}(\Sigma_n \setminus \cup_{k=0}^{\infty} \sigma^{-k}(B_i)) = \Sigma_n \setminus \cup_{i=0}^{\infty} \cup_{k=0}^{\infty} \sigma^{-k}(B_i)$. $\square$

Theorem 5.10. *If $K \subseteq \Sigma_n$ and $\sigma_{|K}$ is Devaney chaotic, then $\omega_\sigma : K \to \mathcal{K}(K)$ is nowhere continuous.*

Proof. Let $y \in P(\sigma_{|K})$; say $y = \overline{y_0 y_1 \ldots y_{p-1}}$. Let $x \in K$. Take $z \in B_{\frac{\epsilon}{2}}(x)$ such that $\omega(z, \sigma) = K$. Since $y \in \omega(z, \sigma)$, $\gamma(z, \sigma)$, the trajectory of z under σ, visits the p-block $B_{y_0 \ldots y_{p-1}}$ infinitely often.
Take k such that $\frac{1}{n^k} < \frac{\epsilon}{2}$. There exists $q > k$ so that $z = z_0 \ldots z_q y_0 y_1 \ldots y_{p-1} \ldots$. Consider $z' = z_0 \ldots z_q \overline{y_0 y_1 \ldots y_{p-1}}$. Then $z' \in B_\epsilon(x)$ and $\omega(z', \sigma) = \omega(y, \sigma)$. It follows that $\{z \in K : \omega(z, \sigma) = \omega(y, \sigma)\}$ is dense. Since $\mathcal{S}_{\sigma,K}$ is also dense, ω_σ is everywhere discontinuous. $\square$

6. Examples

Example 6.1. [9] *There exists a continuous function $f : I \to I$ so that f is Li-Yorke chaotic, and $\omega_f : I \to \mathcal{K}(I)$ is a Baire 1 function. Let X be an infinite ω-limit set of f, necessarily a Cantor set, for which the associated simple system has non-empty interior. Then $f_{|X}$ is Li-Yorke chaotic, $\omega_f : X \to \mathcal{K}(X)$ is constant as $\omega(x, f) = X$ for all x in X, and $h(f_{|X}) = 0$.*

Example 6.2. *Let $F : \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}} \to \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}}$ be as in Theorem 3.2 and let $A : 2^{\mathbb{N}} \to 2^{\mathbb{N}}$ be the adding machine; that is, $A(\alpha_1, \alpha_2, \dots) = (\alpha_1, \alpha_2, \dots) + (1, 0, 0, \dots)$, where the addition is not intended coordinatewise but with "carry over" to the right. Let $X = \{0, \dots, n-1\} \times (2^{\mathbb{N}})^{\mathbb{N}} \times 2^{\mathbb{N}}$ and let $f = F \times A$. As F has infinite topological entropy so does f. Moreover, f is not Devaney chaotic as it has no periodic points, since A has none. Moreover ω_f is everywhere discontinuous as the following two sets are dense*

$$B_1 = \{z \in X : \omega_f(z) = X\}$$

and

$$B_2 = \{z \in X : \omega_f(z) = \{0, 1, \dots, n-1\} \times \{\overline{0}\}^{\mathbb{N}} \times 2^{\mathbb{N}}$$

We conclude that $h(T) > 0$ and ω_T everywhere discontinuous are not sufficient to insure that $(2^{\mathbb{N}}, T)$ is Devaney chaotic on a subsystem.

Example 6.3. [17] *There exists a topological dynamical system $(2^{\mathbb{N}}, T)$ such that $h(T) > 0$, and $(2^{\mathbb{N}}, T)$ is minimal, so that $\omega_T : 2^{\mathbb{N}} \to \mathcal{K}(2^{\mathbb{N}})$ is constant. We conclude that for a topological dynamical system $(2^{\mathbb{N}}, f)$, $h(f) > 0$ does not imply that $\omega_f : 2^{\mathbb{N}} \to \mathcal{K}$ must be somewhere discontinuous.*

Example 6.4. [16] *Let (Σ_2, σ) be the topological dynamical system given by the shift map σ on Σ_2, the space of sequences generated by two symbols. There exists $(K, \sigma) \subseteq (\Sigma_2, \sigma)$ so that $h(\sigma_{|K}) = 0$ and $\sigma_{|K}$ is Devaney chaotic. From Theorem 5.10 we see that $\omega_{\sigma_{|K}}$ is everywhere discontinuous. We conclude that for a topological dynamical system $(2^{\mathbb{N}}, f)$, the map $\omega_f : 2^{\mathbb{N}} \to K$ everywhere discontinuous does not imply $h(f) > 0$.*

Theorem 6.5. [9] *If $f \in C(I, I)$ and $\omega_f : I \to \mathcal{K}$ is continuous, then f is a 2^0-function or a 2^1-function.*

Since positive topological entropy implies Li-Yorke chaos, Example 6.3 shows that Theorem 6.5 due to Bruckner and Ceder does not hold for an arbitrary topological dynamical system $(2^{\mathbb{N}}, f)$.

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